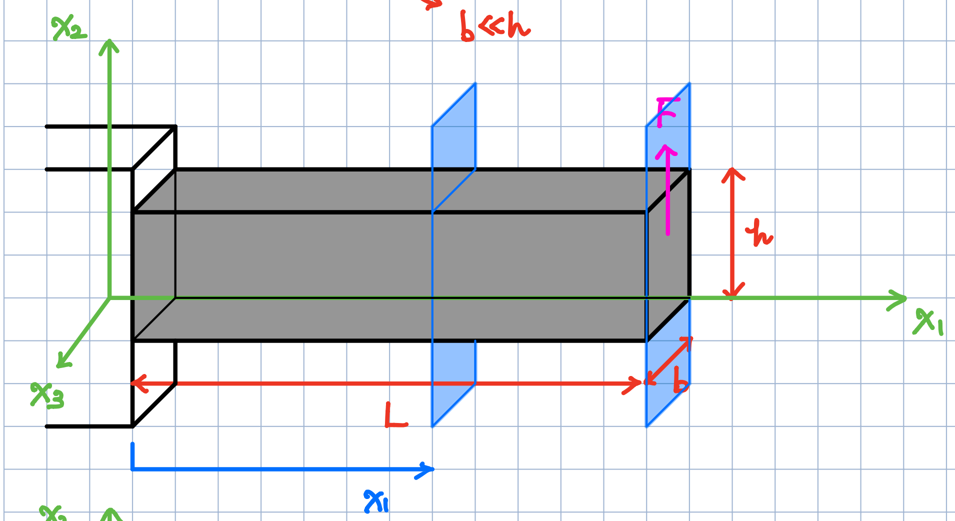
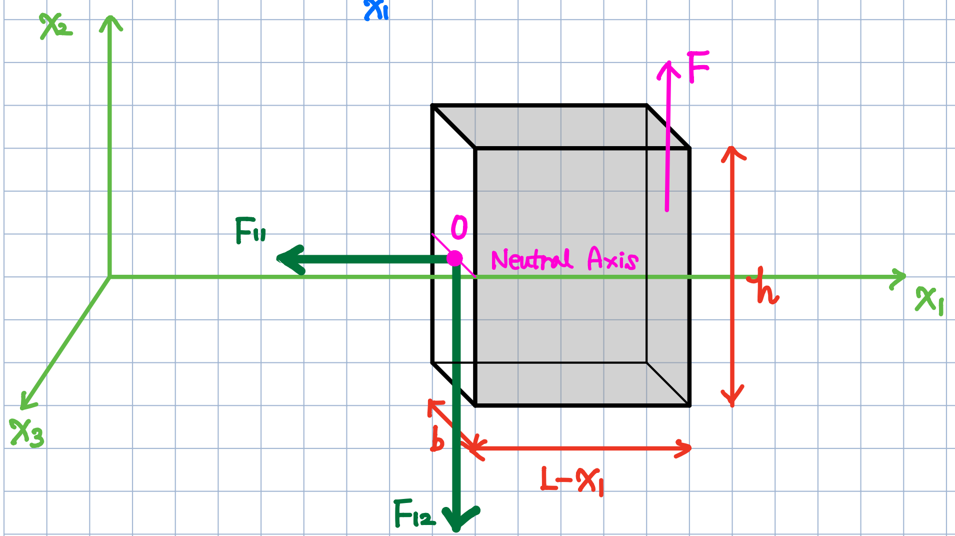
Bending of an slender elastic beam

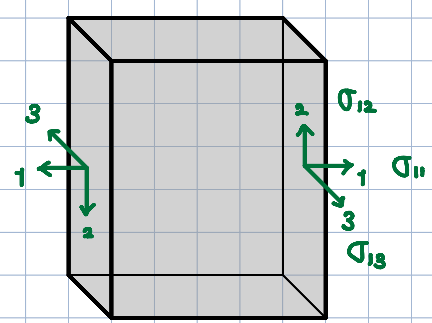
$b \ll h$



• 3 aspects

1. Equilibrium : Satisfy equation
2. Strain & displacement
3. Stress & strain





Step 1 : equilibrium

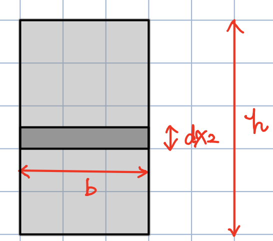
Let's satisfy equilibrium !

$\Sigma F_{x_1} = 0$; $-F_{11} = 0 \longrightarrow F_{11} = 0 \longleftarrow \sigma = F/A$, $F = \sigma \cdot A$

$$\int_A \sigma_{11}\, dA = 0$$

$$\int_{-h/2}^{h/2} \sigma_{11}\, b \cdot dx_2 = 0$$

$$\therefore\ b \cdot \int_{-h/2}^{h/2} \sigma_{11}\, dx_2 = 0$$



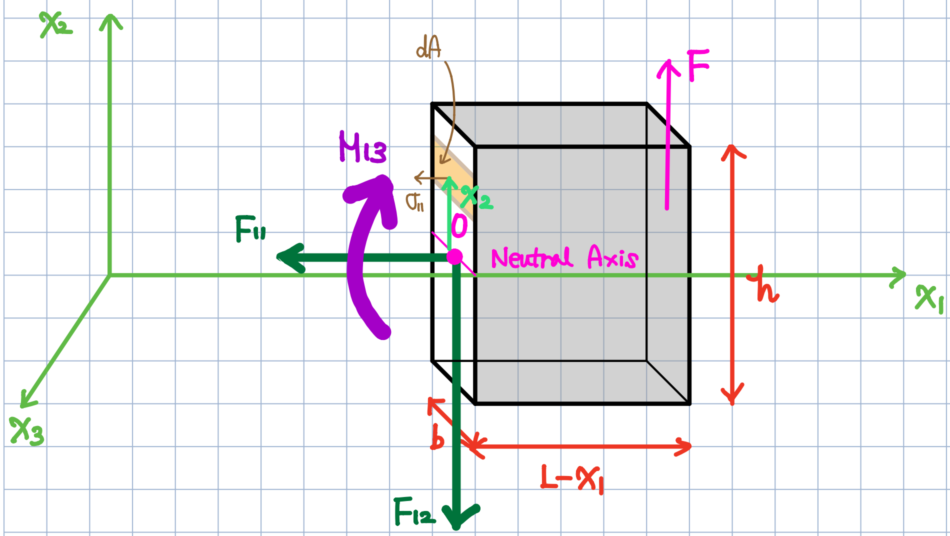
$A = b \cdot h$

$dA = b \cdot dx_2$

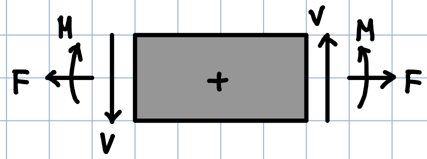
$= \int_A dA = \int_{-h/2}^{h/2} b \cdot dx_2$

$\Sigma F_{x_2} = 0$; $-F_{12} + F = 0 \longrightarrow F_{12} = F$

$$\int_A \sigma_{12}\, dA = b \cdot \int_{-h/2}^{h/2} \sigma_{12}\, dx_2 = F$$



Sign convention



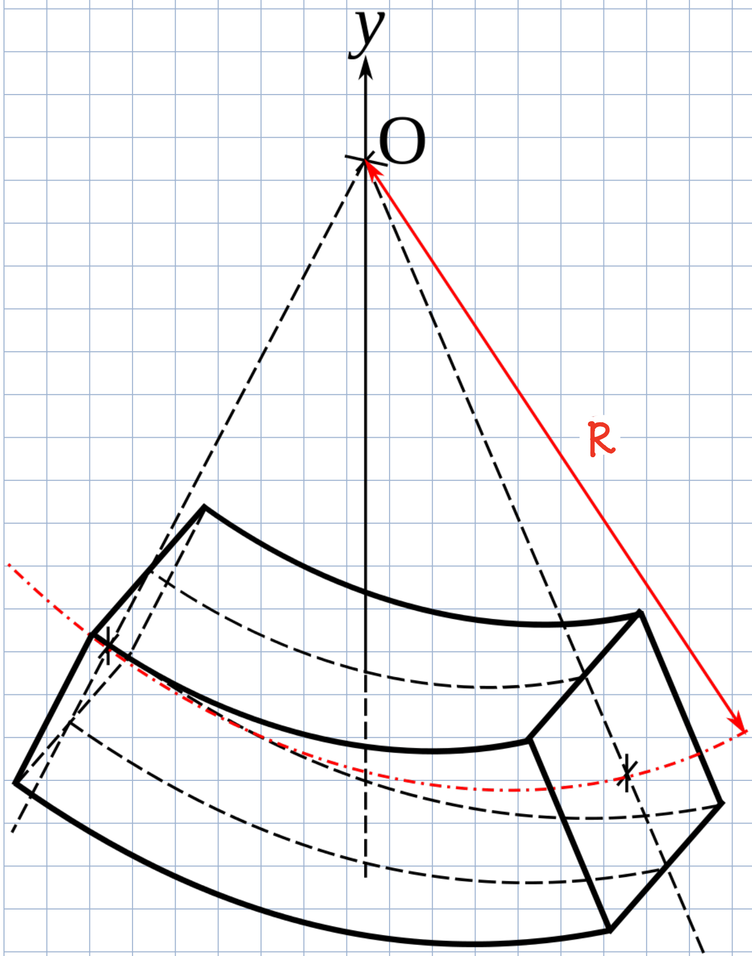
$\Sigma M_{x_3} = 0$; $\quad -M_{13} + F(L-x_1) = 0 \rightarrow M_{13} = F(L-x_1)$

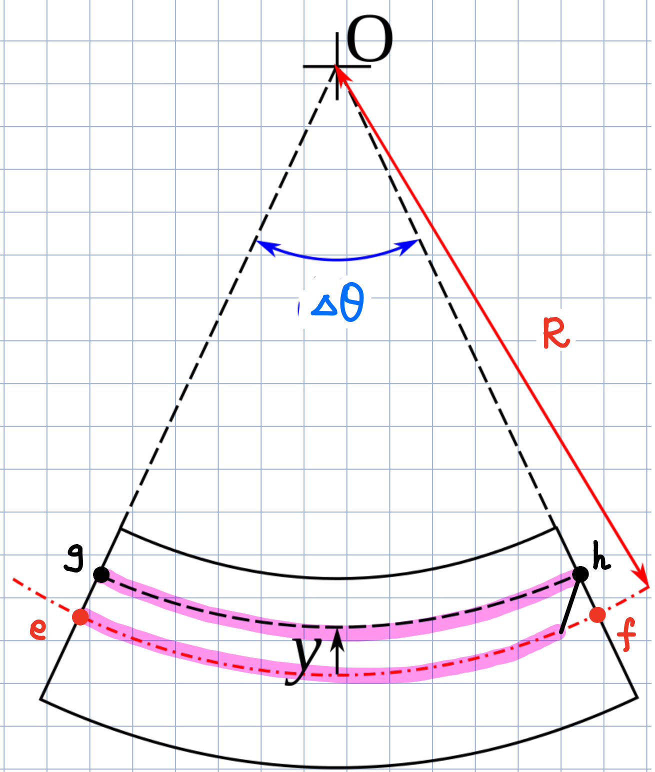
$$F(L-x_1) + \int_A \sigma_{11} x_2 \, dA = 0$$

$$F(L-x_1) = -\int_A \sigma_{11} x_2 \, dA = -b\int_{-h/2}^{h/2} \sigma_{11} x_2 \, dx_2$$

- We don't have to integrate through x_3 (∵ it is slender!)

$$\boxed{\begin{aligned} F_{11} &= b \cdot \int_{-h/2}^{h/2} \sigma_{11} \, dx_2 \\ F_{12} &= b \cdot \int_{-h/2}^{h/2} \sigma_{12} \, dx_2 \\ M_{13} = F(L-x_1) &= -b\int_{-h/2}^{h/2} \sigma_{11} x_2 \, dx_2 \end{aligned}}$$ ***





$$\varepsilon_{11} = \frac{\text{final length} - \text{initial length}}{\text{initial length}} = \frac{(R-y)\Delta\theta \ - R\Delta\theta}{R\Delta\theta}$$

$$= \frac{-y}{R} = -x_2/R$$

Step 2

: Strain & displacement

$$\boxed{\varepsilon_{11} = \frac{-x_2}{R}}$$ ★★★

Constitutive Law

Step3.

1. $\sigma_{11}/\varepsilon_{11} = E_{11}$ (Young's Modulus)

$\downarrow$

$\sigma_{11} = E_{11}\varepsilon_{11} \leftarrow \boxed{\varepsilon_{11} = \frac{-x_2}{R}}$ ★★★

$\downarrow$

$\boxed{\sigma_{11} = -E_{11}\, x_2/R}$

2. $M_{13} = F(L-x_1) = -b\int_{-h/2}^{h/2} \sigma_{11}\, x_2\, dx_2 = b\int_{-h/2}^{h/2} E/R\; x_2^2\, dx_2$

$= E/R \underbrace{\left\{ b\int_{-h/2}^{h/2} x_2^2\, dx_2 \right\}}$

I_{22} = the second moment of inertia

$\int_A x_2^2\, dA = bh^3/12$

$\therefore M_{13} = E/R\; I_{22} \longleftarrow \boxed{\sigma_{11} = E\varepsilon_{11}} \longrightarrow E = \sigma_{11}/\varepsilon_{11} \leftarrow \boxed{\varepsilon_{11} = \frac{-x_2}{R}}$ ★★★

$E/R = M_{13}/I_{22}$

$E = \sigma_{11}\cdot R/-x_2$

$\therefore \sigma_{11} = -M_{13}x_2/I_{22}$ Ans.

$\therefore E/R = \sigma_{11}/-x_2$

$1/R = M_{13}/EI_{22}$

Small deflections : $\boxed{1/R \approx d^2u_2/dx_1^2} = M_{13}/EI_{22} = \frac{F(L-x_1)}{EI_{22}}$

integrate

$du_2/dx_1 = F/EI_{22}\,(Lx_1 - x_1^2/2) + \cancel{C_1}$ 0 ∵BC2

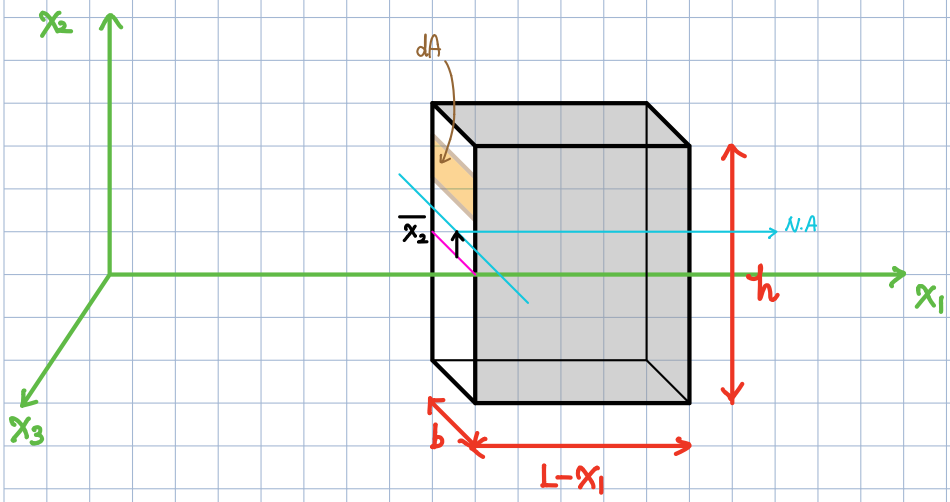
B.C. at $x_1=0 \rightarrow u_2 = 0$

$du_2/dx_1 = 0$

$\therefore u_2 = F/EI_{22}\,(Lx_1^2/2 - x_1^3/6) + \cancel{C_2}$ 0 ∵BC1

Ans.

How can we find N.A.?



$$\boxed{\varepsilon_{11} = \frac{-x_2}{R}}$$ ★★★

$$\boxed{M_{13} = -b\int_{-h/2}^{h/2} \sigma_{11}\, x_2\, dx_2}$$ ★★★

$$\varepsilon_{11} = -\frac{(x_2-\bar{x}_2)}{R}$$

$$M_{13} = -b\int_{-h/2}^{h/2} \sigma_{11}\,(x_2-\bar{x}_2)\,dx_2$$

$$F_{11} = b\cdot\int_{-h/2}^{h/2} \sigma_{11}\, dx_2 = 0 \longleftarrow \sigma_{11} = E\varepsilon_{11} = \frac{-E(x_2-\bar{x}_2)}{R}$$

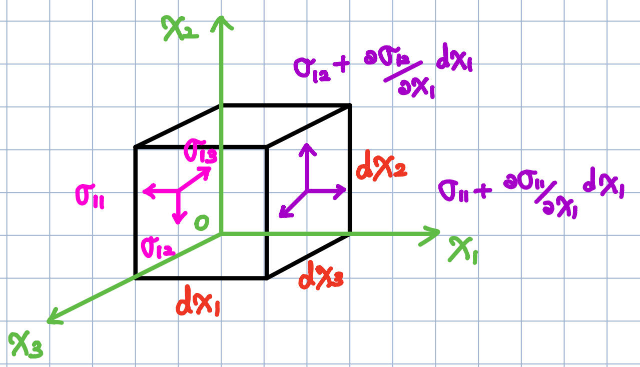
$$= \frac{-bE}{R}\int_{-h/2}^{h/2} (x_2-\bar{x}_2)\,dx_2 = 0$$

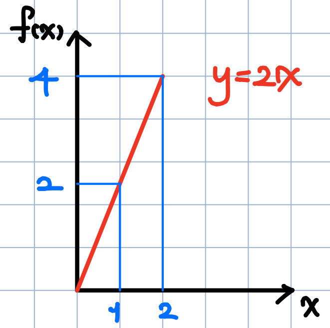
We can find where the Neutral Axis!

Equilibrium Eqns

infinitessimal element

$F = m\cancel{a} = 0$ ∵ statics





$f_{(1)} = 2(1) = 2$

$f_{(2)} = 2(2) = 4$

$f_{(2)} = f_{(1)} + \frac{df_{(x)}}{dx} \cdot \Delta x$

$= 2 + (2)\cdot(2-1) = 2+2 = 4$

$$\Sigma f_{x_1} = \left\{\left(\sigma_{11} + \partial\sigma_{11}/\partial x_1 \, dx_1\right) - \sigma_{11}\right\} dx_2 dx_3 +$$
$$\left\{\left(\sigma_{21} + \partial\sigma_{21}/\partial x_2 \, dx_2\right) - \sigma_{21}\right\} dx_1 dx_3 +$$
$$\left\{\left(\sigma_{31} + \partial\sigma_{31}/\partial x_3 \, dx_3\right) - \sigma_{31}\right\} dx_2 dx_1 + X_F$$

body force : gravitational, magnetic, ...

$$= \rho \, dx_1 dx_2 dx_3 \, \cancel{a_1}$$

0 ∵ statics

$\rightarrow \sigma_{ij} = \sigma_{ji}$

$$\boxed{\begin{aligned} &\partial\sigma_{11}/\partial x_1 + \partial\sigma_{12}/\partial x_2 + \partial\sigma_{13}/\partial x_3 = 0 \\ &\partial\sigma_{21}/\partial x_1 + \partial\sigma_{22}/\partial x_2 + \partial\sigma_{23}/\partial x_3 = 0 \\ &\partial\sigma_{31}/\partial x_1 + \partial\sigma_{32}/\partial x_2 + \partial\sigma_{33}/\partial x_3 = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} F_{11} &= b \cdot \int_{-h/2}^{h/2} \sigma_{11} \, dx_2 \\ F_{12} &= b \cdot \int_{-h/2}^{h/2} \sigma_{12} \, dx_2 \\ M_{13} = F(L-x_1) &= -b \int_{-h/2}^{h/2} \sigma_{11} \, x_2 \, dx_2 \end{aligned}}$$ ***

$$F_{12} = F = b \int_{-h/2}^{h/2} \sigma_{12} \, dx_2 \longleftarrow \sigma_{11} = -M_{13} x_2 / I_{22} = -F(L-x_1) x_2 / I_{22}$$

$$\sigma_{11} = -F/I_{22} \, (L - x_1) x_2$$

$$\boxed{\partial\sigma_{11}/\partial x_1 = -\partial\sigma_{12}/\partial x_2}$$

∵ slender

$$\therefore \partial\sigma_{11}/\partial x_1 = F x_2 / I_{22} = -\partial\sigma_{12}/\partial x_2 \longrightarrow \sigma_{12} = -F x_2^2 / 2I_{22} + f_n(x_1, \cancel{x_3})$$

$$\partial\sigma_{12}/\partial x_1 = -\cancel{\partial\sigma_{22}/\partial x_2}$$ $\sigma_{22}=0$, every where

$$\sigma_{12} = -F x_2^2/2I_{22} + C \longleftarrow \text{B.C.}$$

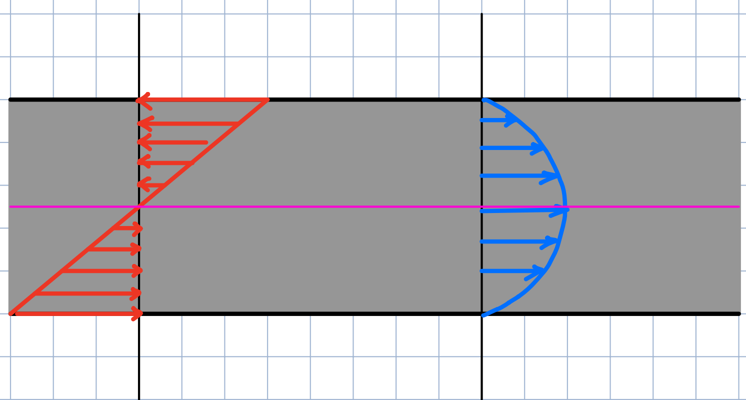
$\sigma_{12} = 0$ at free surface

when $x_2 = \pm h/2$

$$\therefore C = F/2I_{22}\,(h^2/4)$$

$$\therefore \sigma_{12} = F/2I_{22}\,(h^2/4 - x_2^2)$$

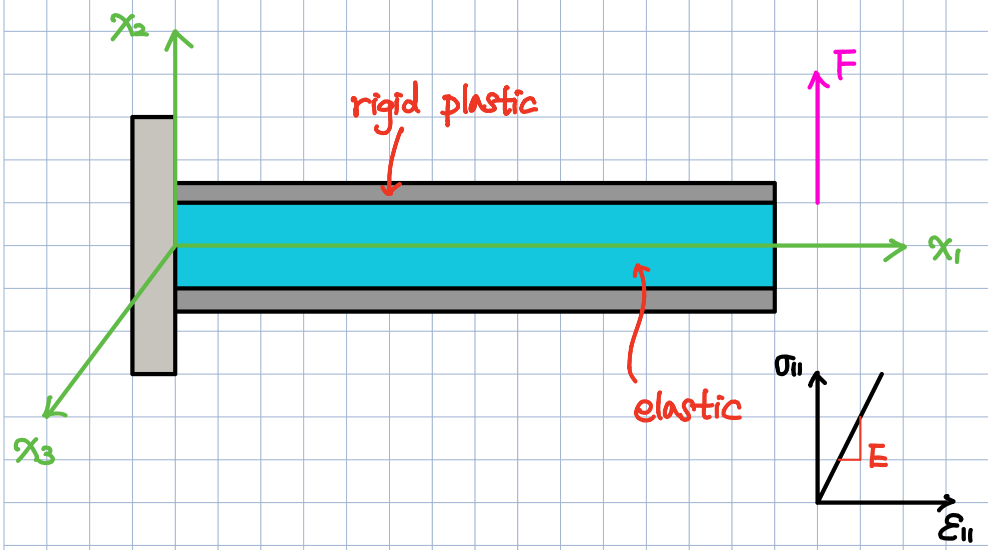
Ans.



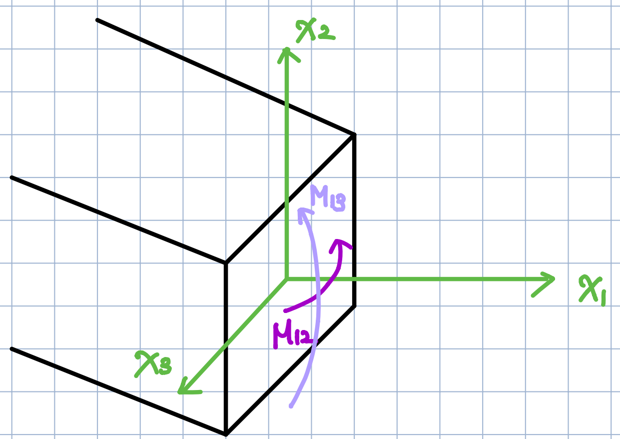
$$\boxed{\sigma_{11} = -M_{13}x_2/I_{22}}$$

$$\boxed{\sigma_{12} = F/2I_{22}\,(h^2/4 - x_2^2)}$$

Composite Beam (Slender)



multiaxis loading



Governing Equations are linear!

Principle of superposition.

$f(x_1+x_2) = f(x_1) + f(x_2)$

M13 only : $\sigma_{11} = -M_{13}x_2 / I_{22} \quad \longleftarrow \quad I_{22} = bh^3/12$

M12 only : $\sigma_{11} = M_{12}x_3 / I_{33} \quad \longleftarrow \quad I_{33} = hb^3/12$

M12 & M13 : $\sigma_{11} = \dfrac{M_{12}x_3}{I_{33}} - \dfrac{M_{13}x_2}{I_{22}}$ Ans.