Select the best material for each of the requirements.

| Property | Material A | Material B | Material C |
|---|---|---|---|
| Elastic modulus (ksi) | 10,000 | 30,000 | 40,000 |
| Yield strength (ksi) | 30 | 300 | – |
| Ultimate strength (ksi) | 50 | 320 | 200 |
| True fracture strength (ksi) | 100 | 350 | 200 |
| Strain hardening exponent | 0.26 | 0.05 | – |
| Reduction in area (%) | 65 | 15 | 0.01 |
| Density (lb/in$^3$) | 0.28 | 0.25 | 0.40 |

(a) Max. tensile load $\rightarrow$ $\sigma_{UTS}$

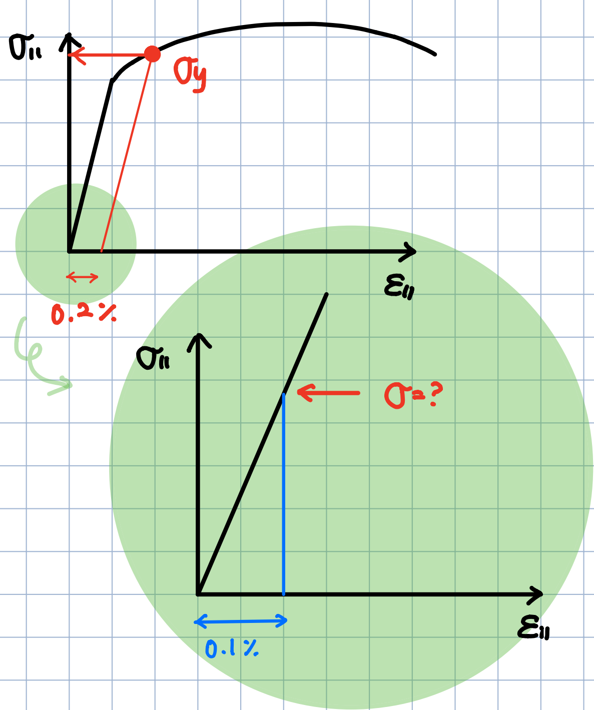
| | $\sigma_{UTS}$ | |
|---|---|---|
| A | 50 | |
| B | 320 | ✓ |
| C | 200 | |

(b) Max. tensile load per pound on similar rods $\rightarrow$ $\sigma_{UTS}$/density

| | $\sigma_{UTS}$ | density | $\sigma_{UTS}$/density | |
|---|---|---|---|---|
| A | 50 | 0.28 | 178.6 | |
| B | 320 | 0.25 | 640 | ✓ |
| C | 200 | 0.40 | 500 | |

(c) Max. tensile load at $e = 0.001$

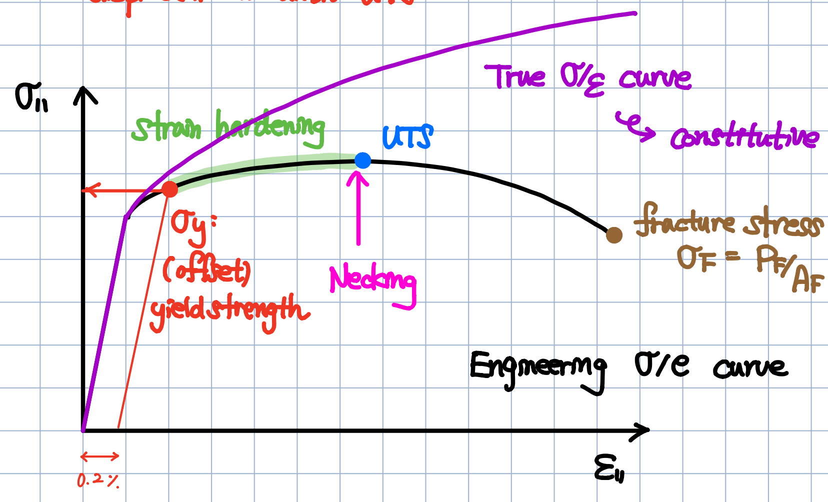

| Property | Material A | Material B | Material C |
|---|---|---|---|
| Elastic modulus (ksi) | 10,000 | 30,000 | 40,000 |
| Yield strength (ksi) | 30 | 300 | – |
| Ultimate strength (ksi) | 50 | 320 | 200 |
| True fracture strength (ksi) | 100 | 350 | 200 |
| Strain hardening exponent | 0.26 | 0.05 | – |
| Reduction in area (%) | 65 | 15 | 0.01 |
| Density (lb/in$^3$) | 0.28 | 0.25 | 0.40 |


| | Elastic modulus | |
|---|---|---|
| A | 10,000 | |
| B | 30,000 | |
| C | 40,000 | ✓ |

## (d) Max. uniform elongation before necking

= displacement until UTS




↳ constitutive law : $\sigma_{11} = k(\varepsilon_{11})^n$, n: strain hardening exponent

• plastic deformation : $\Delta V/V = 0$

$\nu = \frac{1}{2}$

$V = A\cdot l = A_0\cdot l_0 \rightarrow l/l_0 = A_0/A$

$dV = 0 = d(A\cdot l)$

$= dA\cdot l + A\cdot dl = 0$

$dl/l = -dA/A$

Engineering $\sigma^e/e$ ⟷ True $\sigma^T/\varepsilon$

$\sigma_{11}^e = P/A_0$

$e_{11} = \Delta l/l_0$

$\sigma_{11}^T = P/A = P/A_0 \cdot A_0/A = \sigma_{11}^e \cdot l/l_0 = \sigma_{11}^e\cdot(1+e_{11})$

$$d\varepsilon_{11} = dl/l \rightarrow \varepsilon_{11} = \int_{l_0}^{l} d\varepsilon_{11}$$

$$= \int_{l_0}^{l} dl/l$$

$$= \ln l/l_0 = \ln \frac{l_0 + \Delta l}{l_0}$$

$$= \ln(1+e_{11})$$

$\varepsilon_{11} = \ln(1+e_{11})$

$e_{11}$ is too small,

$e_{11} \approx \varepsilon_{11}$

(elastic strain ≈ true strain)

## (d) Max. uniform elongation before necking

= displacement until UTS

$\sigma_{11}^T = P/A \longrightarrow P = \sigma^T\cdot A$ ⟵ $dP=0$ at necking

$$dP = d\sigma^T\cdot A + \sigma^T\cdot dA = 0$$

$$\boxed{-dA/A = d\sigma^T/\sigma^T}$$

$dV = 0 = d(A\cdot l)$

$= dA\cdot l + A\cdot dl = 0$

$\boxed{dl/l = -dA/A}$

$$d\sigma^T/\sigma^T = dl/l = d\varepsilon_{11}$$

$$d\sigma^T/d\varepsilon_{11} = \sigma^T$$ ⟵ constitutive law : $\sigma_{11} = k(\varepsilon_{11})^n$

$$\frac{d}{d\varepsilon_{11}}(\sigma^T) = \sigma^T$$

$$\cancel{k}\cdot n\cdot \varepsilon_{11}^{n-1} = \cancel{k}\cdot \varepsilon_{11}^n$$

$\therefore n = \varepsilon_{UTS}$ Ans. at necking ! why? $dP=0$ **

before necking : $\varepsilon = \ln(1+e)$

$$\rightarrow (1+e) = \exp(\varepsilon)$$

$$\therefore e = \exp(\varepsilon) - 1$$ Ans.

| Property | Material | | |
|---|---|---|---|
| | A | B | C |
| Elastic modulus (ksi) | 10,000 | 30,000 | 40,000 |
| Yield strength (ksi) | 30 | 300 | – |
| Ultimate strength (ksi) | 50 | 320 | 200 |
| True fracture strength (ksi) | 100 | 350 | 200 |
| Strain hardening exponent | 0.26 | 0.05 | – |
| Reduction in area (%) | 65 | 15 | 0.01 |
| Density (lb/in$^3$) | 0.28 | 0.25 | 0.40 |

| | strain hardening exponent | e | |
|---|---|---|---|
| A | 0.26 | 0.2969 | ✓ |
| B | 0.05 | 0.0513 | |
| C | N/A | N/A | |

no necking (brittle)

(e) Max. work to necking

Let's assume ductility dominates material's behavior

$$U = U^T = \int_0^{\varepsilon} \sigma^T d\varepsilon \leftarrow \sigma^T = k\varepsilon^n$$

$$= \int_0^{\varepsilon} k\varepsilon^n d\varepsilon$$

$$= \frac{k}{n+1} \varepsilon^{n+1} \Big]_0^{\varepsilon} = \frac{k \cdot \varepsilon^{n+1}}{n+1} = \frac{\sigma^T \cdot \varepsilon}{n+1} \quad **$$

$\varepsilon_{UTS} = n$

$\sigma^T = \sigma_{UTS} \cdot (1+e)$

$\varepsilon_{UTS} = \ln(1+e_{UTS})$

$\exp\{\varepsilon_{UTS}\} = 1 + e_{UTS}$

$e_{UTS} = \exp\{\varepsilon_{UTS}\} - 1 = \exp\{n\} - 1$

$\sigma_{11}^T = P/A = P/A_0 \cdot A_0/A = \sigma_{11}^e \cdot l/l_0 = \sigma_{11}^e \cdot (1+e_{11})$

$d\varepsilon_{11} = dl/l \rightarrow \varepsilon_{11} = \int_{l_0}^{l} d\varepsilon_{11}$

$= \int_{l_0}^{l} dl/l$

$= \ln l/l_0 = \ln \dfrac{l_0 + \Delta l}{l_0}$

$= \ln(1+e_{11})$

$\therefore \sigma^T = \sigma_{UTS} \cdot (1+e) = \sigma_{UTS} (\cancel{1} + \exp\{n\} \cancel{-1}) \longrightarrow U_{UTS} = \dfrac{\sigma_{UTS} \cdot \exp\{n\} \cdot n}{n+1}$ Ans.

| | $U_{UTS}$ | |
|---|---|---|
| A: | $\dfrac{50 \cdot e^{0.26} \cdot 0.26}{0.26+1} = 13.38$ | |
| B: | $\dfrac{320 \cdot e^{0.05} \cdot 0.05}{0.05+1} = 16.02$ | ✓ |
| C: | no necking (brittle) | |


| Property | Material | | |
|---|---|---|---|
| | A | B | C |
| Elastic modulus (ksi) | 10,000 | 30,000 | 40,000 |
| Yield strength (ksi) | 30 | 300 | – |
| Ultimate strength (ksi) | 50 | 320 | 200 |
| True fracture strength (ksi) | 100 | 350 | 200 |
| Strain hardening exponent | 0.26 | 0.05 | – |
| Reduction in area (%) | 65 | 15 | 0.01 |
| Density (lb/in$^3$) | 0.28 | 0.25 | 0.40 |

## (f) Max. work (to fracture)

Let's assume ductility dominates material's behavior

$$U_f^T = \frac{\sigma_f^T \cdot \varepsilon_f}{n+1}$$

$$u = u^T = \int_0^{\varepsilon} \sigma^T d\varepsilon \leftarrow \sigma^T = k\varepsilon^n$$
$$= \int_0^{\varepsilon} k\varepsilon^n d\varepsilon$$
$$= \frac{k}{n+1} \varepsilon^{n+1} \Big]_0^{\varepsilon} = \frac{k \cdot \varepsilon^{n+1}}{n+1} = \frac{\sigma^T \cdot \varepsilon}{n+1}$$

$$\varepsilon_f = \ln A_0/A$$

$$\varepsilon_u = \ln l/l_0 = \ln A_0/A$$

$$RA\% = \frac{A_0 - A}{A_0} \times 100 = x$$

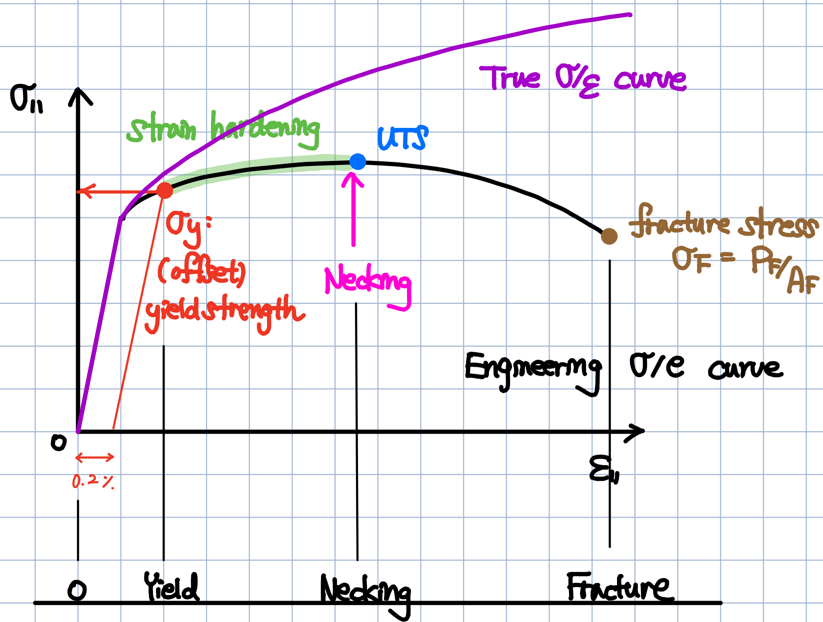
$$\therefore x = \frac{RA\%}{100}$$

$$A_0 - A = A_0 \cdot x$$
$$A_0(1-x) = A$$
$$A_0/A = \frac{1}{1-x}$$

$$\varepsilon_f = \ln A_0/A = \ln\left\{\frac{1}{1-x}\right\}$$

| | Max. work to fracture ($= U_f^T$) |
|---|---|
| A | $\dfrac{(100) \cdot \ln\left\{\frac{1}{1-0.65}\right\}}{0.26+1} = 83.32$ ✓ |
| B | $\dfrac{(350) \cdot \ln\left\{\frac{1}{1-0.15}\right\}}{0.05+1} = 54.17$ |

| Property | Material A | Material B | Material C |
|---|---|---|---|
| Elastic modulus (ksi) | 10,000 | 30,000 | 40,000 |
| Yield strength (ksi) | 30 | 300 | – |
| Ultimate strength (ksi) | 50 | 320 | 200 |
| True fracture strength (ksi) | 100 | 350 | 200 |
| Strain hardening exponent | 0.26 | 0.05 | – |
| Reduction in area (%) | 65 | 15 | 0.01 |
| Density (lb/in$^3$) | 0.28 | 0.25 | 0.40 |




$\sigma = P/A_0$

$e = \Delta l / l_0$

$\sigma^T = P/A$

$\varepsilon = \ln(1+e)$

$\sigma^T = \sigma$ $\quad$ $\sigma^T = \sigma(1+e)$

$\varepsilon^T = \varepsilon$ $\quad$ $\varepsilon^T = \ln A_0/A$

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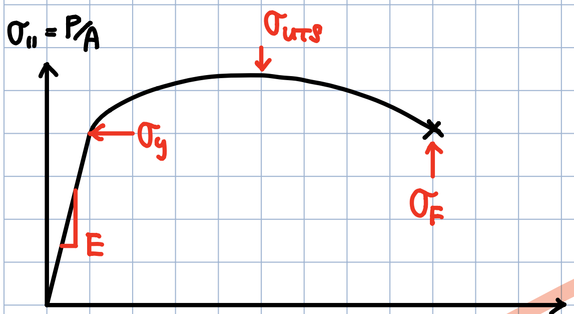
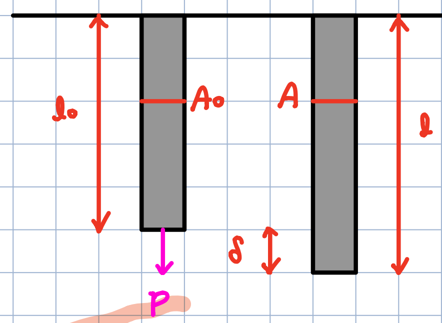
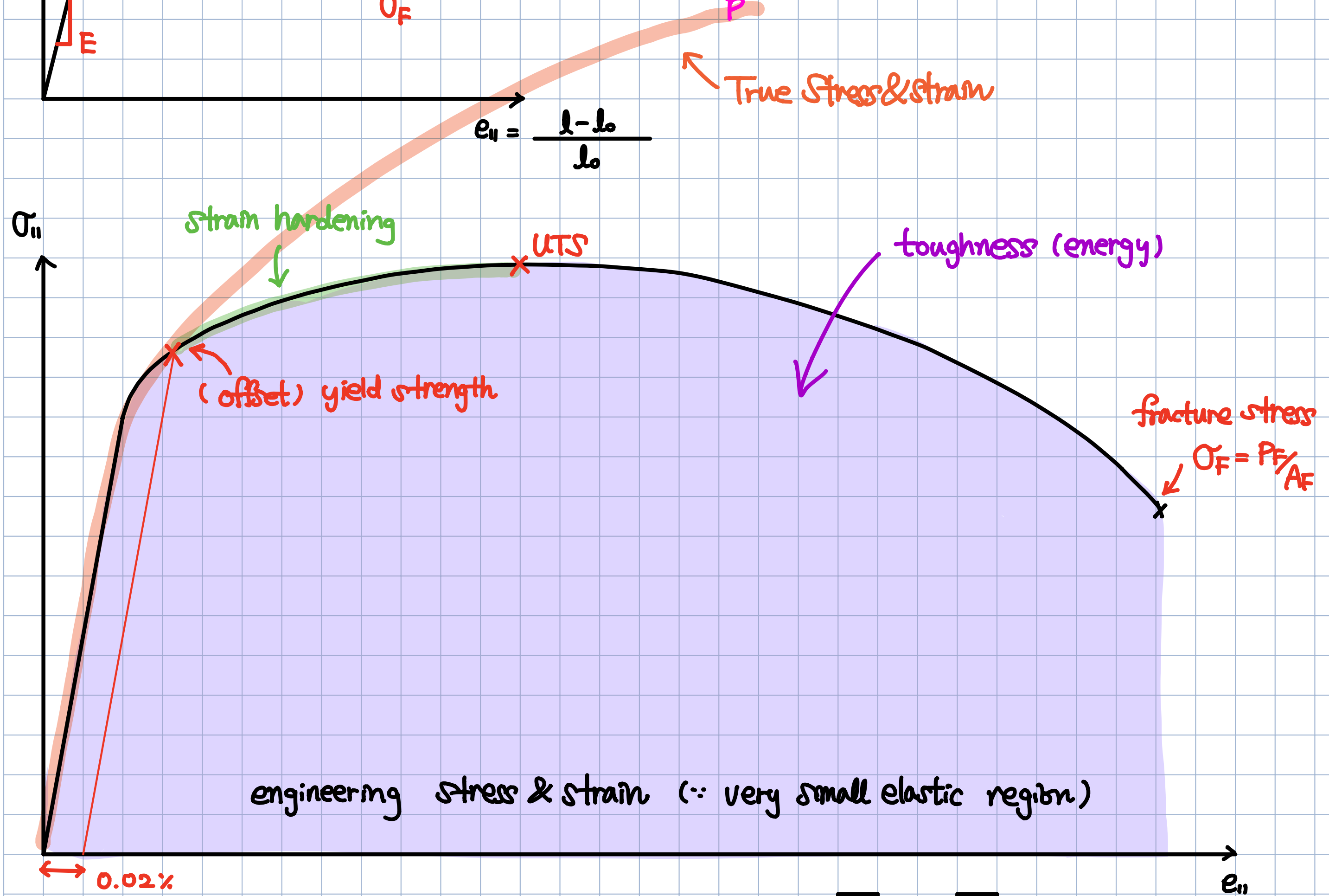
# Plasticity

uniaxial stress-strain curve



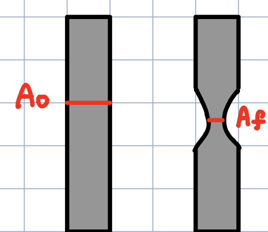




Cu ~ 30-100%

- **stiffness** : elastic modulus

  resistance to plastic deformation
- **strength** : yield stress

  UTS (tensile strength) = $P_{max}/A_0$

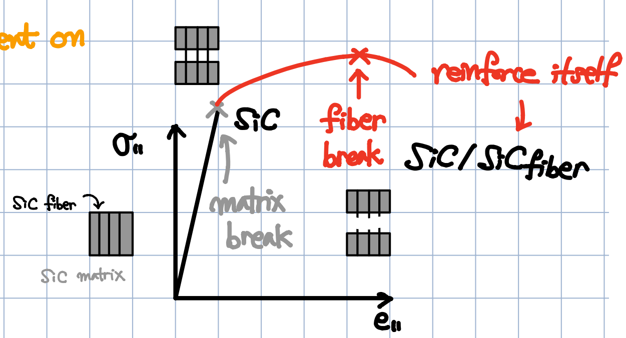



- **ductility** : % elongation (total elongation = strain at failure)

  $= \frac{l_f - l_0}{l_0} \times 100\%$ ← can be dependent on gauge length

  · % reduction in area $= \frac{A_0 - A_f}{A_0} \times 100\%$

  resistance to fracture
- **toughness** : area of stress-strain curve



(True) Stress : $\sigma_{11}^{T} = P/A$ $\longleftrightarrow$ (Engineering) stress : $\sigma_{11}^{e} = P/A_o$

(True) Strain : $d\varepsilon_{11} = dl/l$ (Engineering) strain : $e_{11} = \Delta l/l_o$

$$\varepsilon_{11} = \int_{l_o}^{l} d\varepsilon_{11} = \int_{l_o}^{l} dl/l = \ln l/l_o$$

$$\partial\, \varepsilon_{ij} = \partial\left\{ \tfrac{1}{2}\left( \partial u_i/\partial x_j + \partial u_j/\partial x_i \right)\right\}$$

$\varepsilon^{elastic} \ll \varepsilon^{plastic}$ plastic deformation

$\longrightarrow$ assume elastic strains are negligible.

plastic deformation occurs without changing in volume. $\rightarrow \Delta V/V = 0$ , $\nu = 1/2$

$V = A\cdot l = A_o \cdot l_o \longrightarrow l/l_o = A_o/A = 1+e$ $\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon = \Delta V/V$

$V = A\cdot l$ , $dV = 0 \longrightarrow A dl + l dA = 0$

$$dl/l = -dA/A = d\varepsilon_{11}$$

$$\sigma_{11}^{T} = P/A = P/A_o \cdot A_o/A = \sigma_{11}^{e}(1+e_{11})$$

$$\varepsilon_{11} = \int_{l_o}^{l} dl/l = \ln l/l_o = \ln(1+e_{11})$$

$\leftarrow$ $x \approx \ln(1+x)$ , when $x$ is small

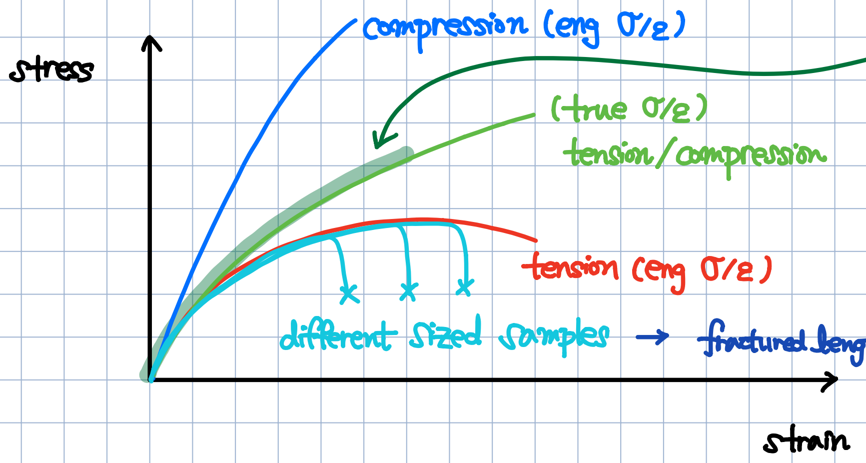
$e_{11}$ is too small, $e_{11} \approx \varepsilon_{11}$

elastic strain $\approx$ plastic strain.

$\longrightarrow$ up to maximum load (necking)

constitutive law for plasticity

$$\sigma_{11} = k\,\varepsilon_{11}^{n}$$




$\varepsilon_{11}^{p} = \int d\varepsilon_{11}^{p}$ (ignore $\varepsilon_{11}^{e}$)

$n$ = strain hardening exponent

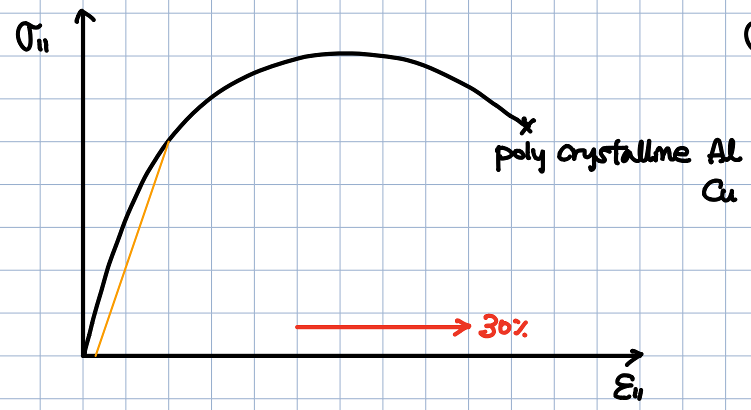
$0 < n < 1$ , $n$ : 0.01–0.2

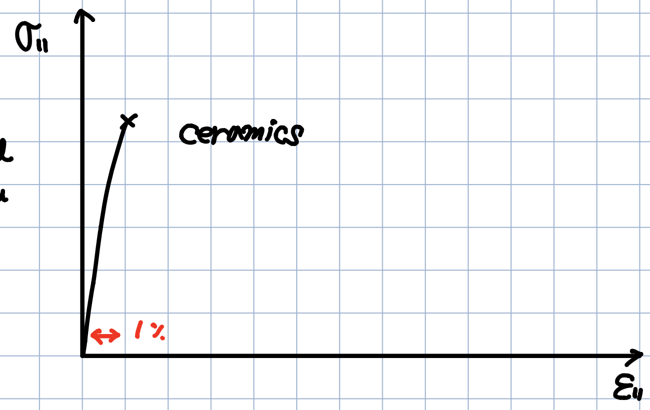
$\rightarrow$ fractured lengths are different by sizes

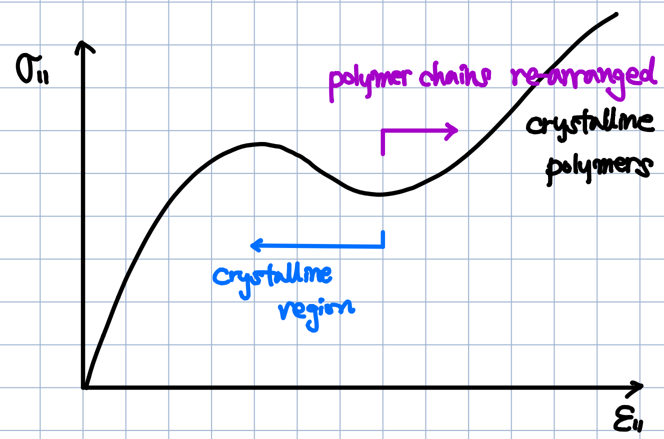
(∵ it depends on the gauge length)

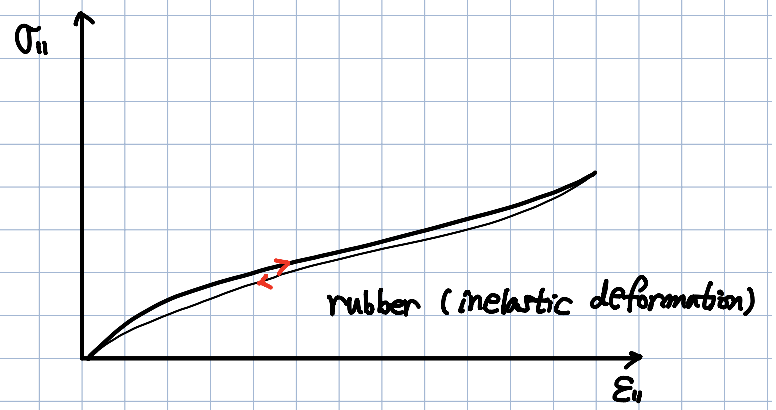
ductility

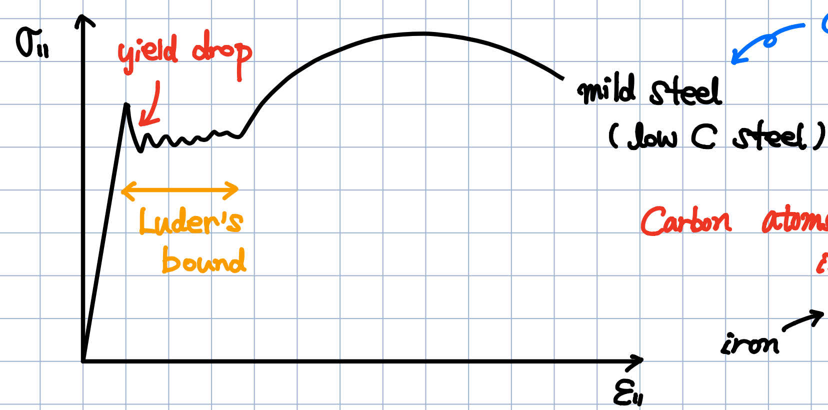
$\varepsilon_{11} = k' \sigma_{11}^{N}$ , $1 < N < \infty$ (mechanical eng.)



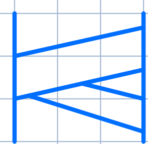



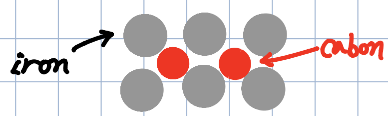



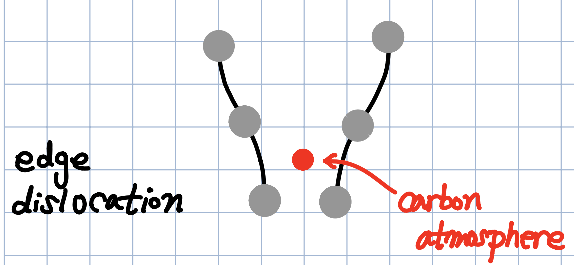




Car bodies : you don't want yield drop

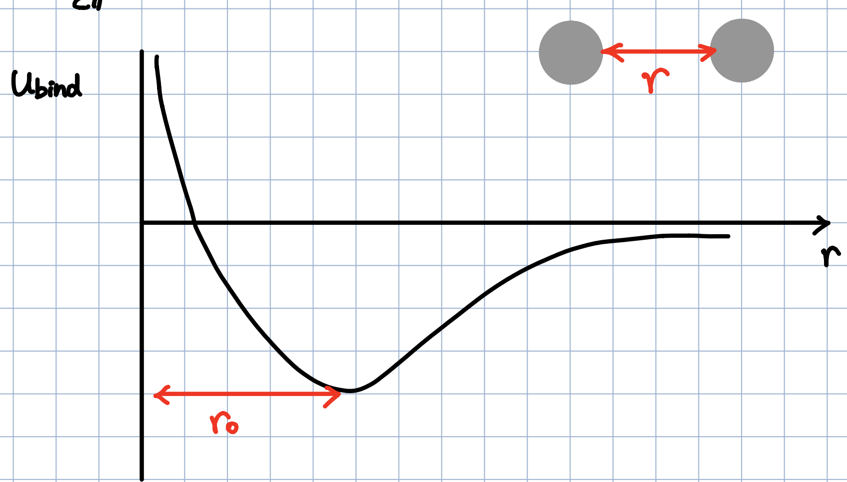


Carbon atoms : very small
interstitial element

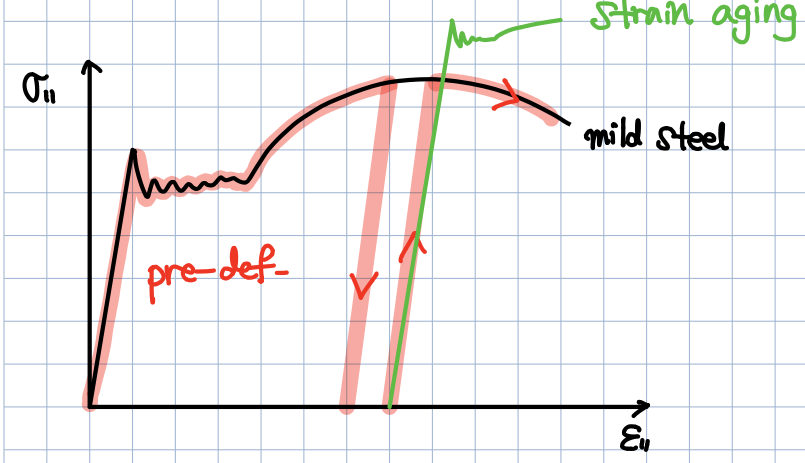


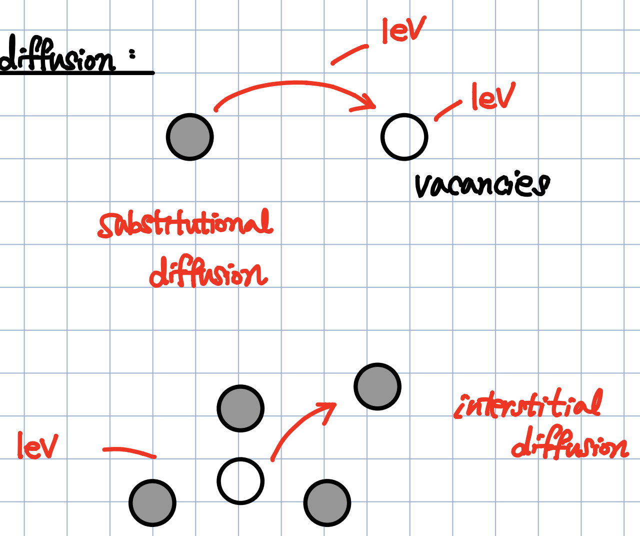

solid solution hardening

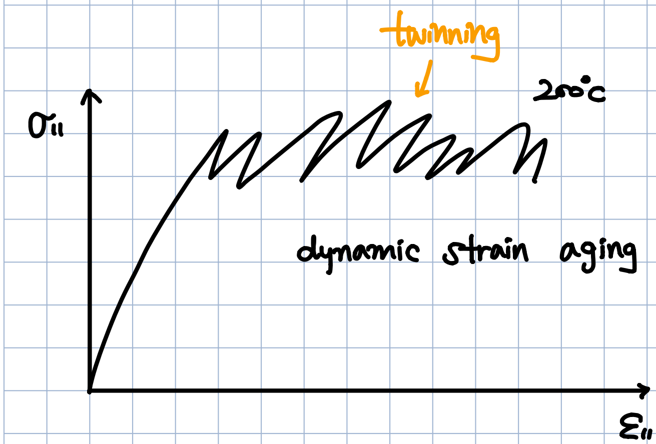


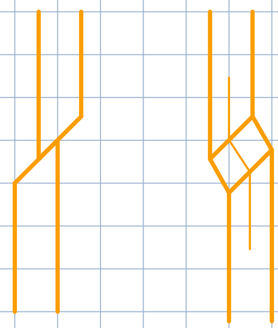
pre-deform the steel (rolling)




diffusion :

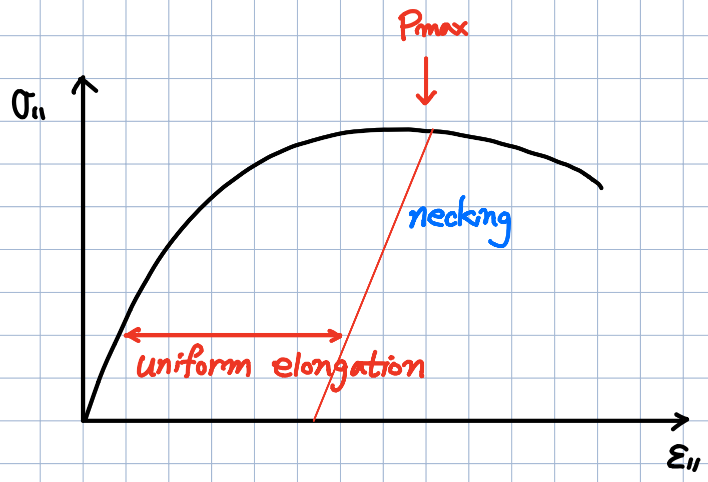


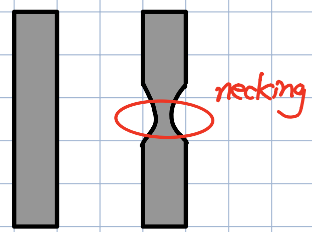



Twin

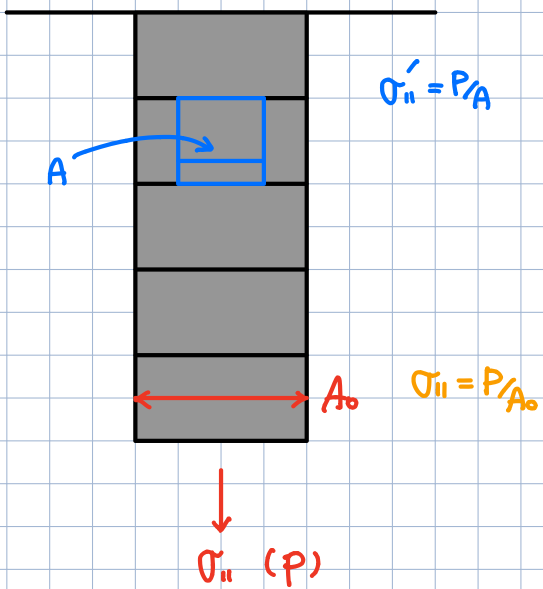
## Necking (Plastic instability)




$e_{11} = \Delta l / l_0$

## Why does necking occur?




(atomic preferences, ⋯ )

for any reasons, A becomes smaller than $A_0$

work hardening

as $A < A_0 \longrightarrow \sigma_{11}' = \sigma_{11}$

geometric softening

uniform elongation ⟶ natural competition between strain hardening & geometric softening

⟶ Necking condition is $dP = 0$

uniaxial tension : $\sigma_{11} = P/A$

$$P = A\sigma_{11}$$

$$dP = A\,d\sigma_{11} + dA\,\sigma_{11} = 0$$

$$\therefore \quad d\sigma_{11}/\sigma_{11} = -dA/A \quad \longleftarrow \quad \varepsilon_{11}^{e} \ll \varepsilon_{11}^{p}, \quad dl/l = -dA/A$$

$\Delta V / V = 0$

$$= dl/l = d\varepsilon_{11}$$

$$\boxed{d\sigma_{11}/d\varepsilon_{11} = \sigma_{11} \quad \text{at necking}}$$ (considering relationship)

$\varepsilon_{11} = n$ at necking

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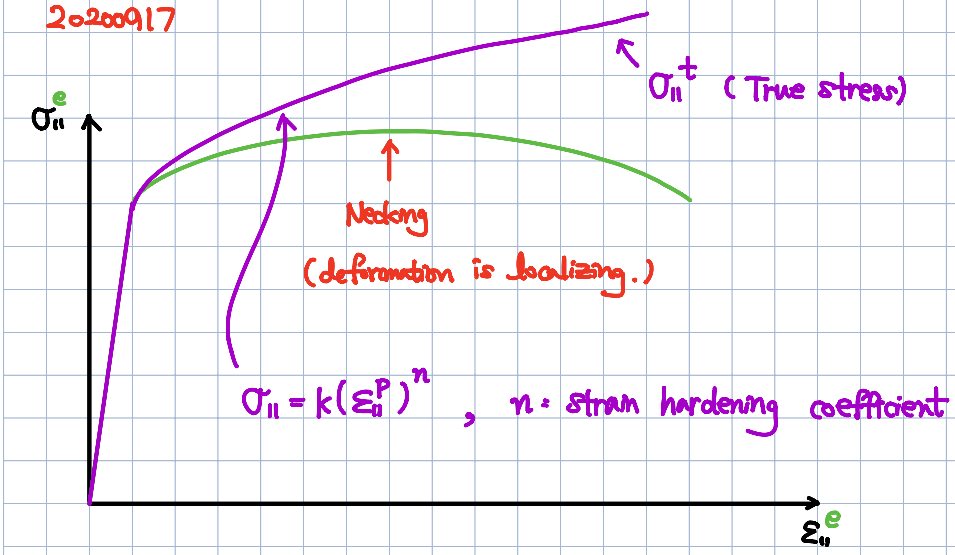


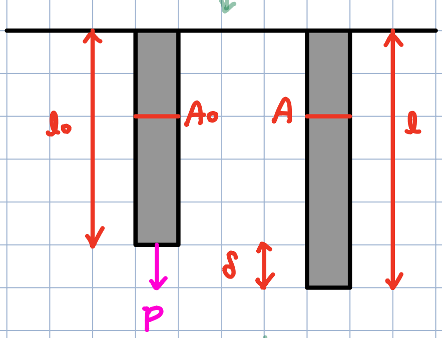

$\sigma_{11}^{e} = P/A_0$

$e_{11} = \Delta l / l_0$

not realistic
∵ it doesn't reflect current situation

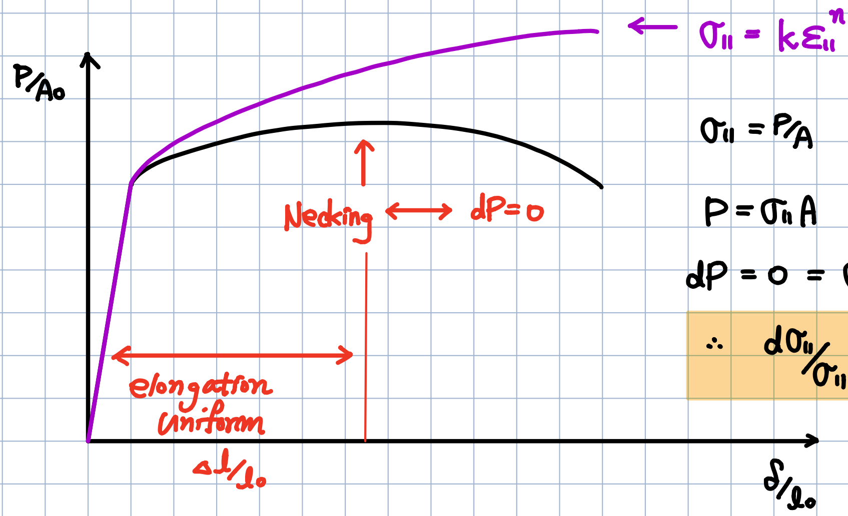
$\sigma_{11} = P/A = \sigma_{11}^{e}(1+e_{11})$

$\varepsilon_{11} = \int d\varepsilon_{11} = \ln l/l_0 = \ln(1+e_{11})$




$\varepsilon_{11}^{e} \ll \varepsilon_{11}^{p} \longrightarrow \Delta V/V = 0 \longrightarrow dl/l = -dA/l = d\varepsilon_{11}$

$l/l_0 = A_0/A = 1 + e_{11}$




$\sigma_{11} = P/A$

$P = \sigma_{11} A$

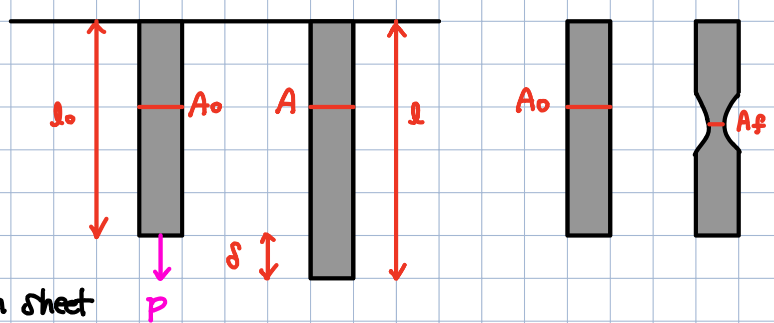
$dP = 0 = \sigma_{11} dA + A d\sigma_{11}$

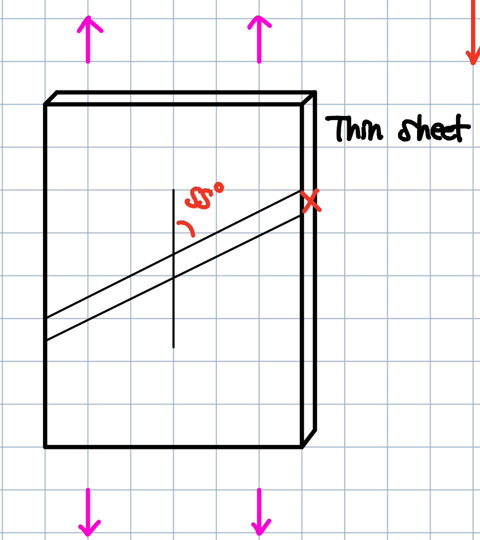
$\therefore \; d\sigma_{11}/\sigma_{11} = -dA/A = dl/l = d\varepsilon_{11}$ at necking

(diffuse necking)

Necking criteria $\longrightarrow$ $\underline{d\sigma_{11}/d\varepsilon_{11} = \sigma_{11}}$ Ans.

(Considère criteria)




necking criteria : $d\sigma_{11}/d\varepsilon_{11} = \sigma_{11}$

$\longrightarrow \; \dfrac{d\sigma_{11}}{d\varepsilon_{11}} = k \cdot n\, \varepsilon_{11}^{n-1} = \sigma_{11} = k \cdot \varepsilon_{11}^n$

constitutive law : $\sigma_{11} = k\varepsilon_{11}^n$

$\underline{\therefore \; \varepsilon_{11} = n \text{ at necking}}$ Ans.

strain-induced $\gamma \rightarrow \alpha'$ (martensite)

TRIP transformation-induced plasticity

TWIP twinning-induced plasticity

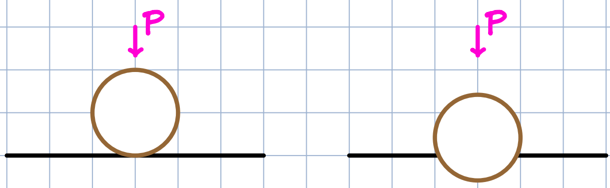
$n \sim 0.2$ low C (mild steel)

$n \sim 0.02$ high C alloy steel $\leftarrow$ high strength

$n \sim 0.5$ austenitic stainless steel 18Cr-8Ni

Hadfield Mn steel 13Mn steel

Hardness $\longleftrightarrow$ Strength



- Brinell hardess = $P/$surface area of indent = stress ($kg/mm^2$) = BHN
  ↑ Ball

hardness (Brinell) ~ 3× UTS

- Meyer hardess = $P/$projected area of indent

- Vicker's hardness : diamond, pyramid indenter

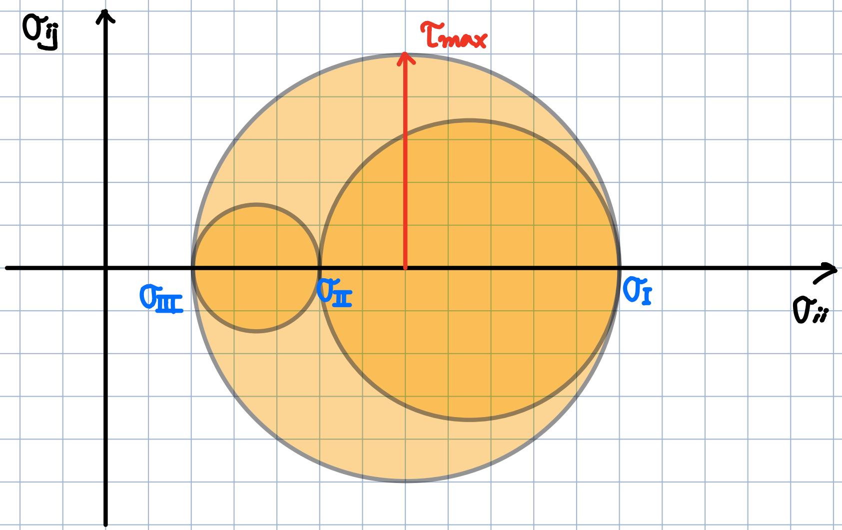
  (cubic corner — nanoindentation)

- Superficial hardness : Rockwell hardness — cone or ball

| Material | $\nu$ | E(GPa) | $\sigma_y$ (MPa) | $\sigma_{UTS}$ (MPa) |
|---|---|---|---|---|
| Mildsteel | 0.3 | 210 | 300 | 420 |
| High strength alloy steel | 0.3 | 210 | 2000 | 2200 |
| Al | 0.3 | 70 | 40 | |
| Al−4%Cu | 0.3 | 70 | 400 | |
| diamond | | 1000 | | |
| Wood | 0.04−0.5 | | | |
| Ceramic $Si_3N_4$ | | 320 | 16−30 | |
| bone | | 15-20 | 100 | |

# Yield Criteria for initial yielding

**Tresca** → initial yielding starts when max. shear stress > shear yield stress ($k$)




$$\tau_{max} = \frac{\sigma_I - \sigma_{III}}{2} = k$$

**Mises** → equivalent stress ($\bar{\sigma}$) = yield strength in tension $y$

$$\bar{\sigma} = \left[ \frac{1}{2} \left\{ (\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 \right\} + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right]^{1/2}$$

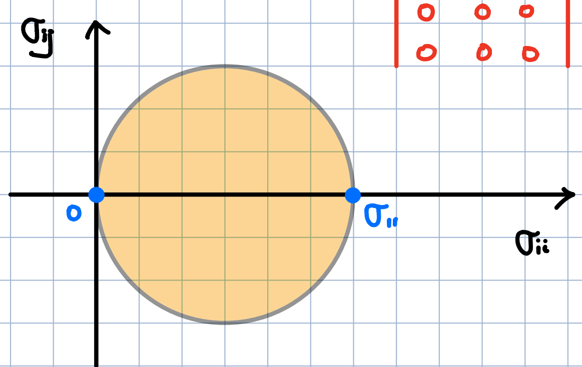
$$= \left[ \frac{1}{2} \left\{ (\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 \right\} \right]^{1/2}$$

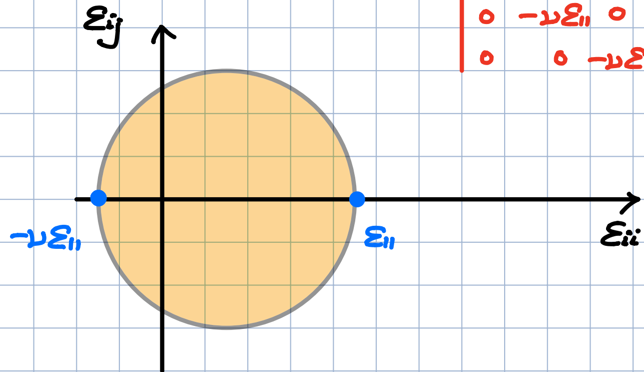
$15\frac{1}{2}\%$ max difference

$$\boxed{\begin{aligned} y &= 2k \quad (\text{Tresca}) \\ &= \sqrt{3}\,k \quad (\text{Mises}) \end{aligned}}$$

**Uniaxial Tension**

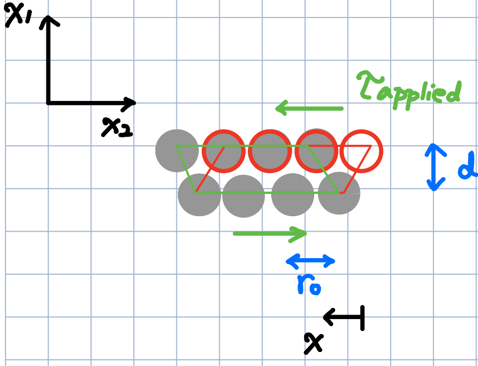
$$\begin{vmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & -\nu\varepsilon_{11} & 0 \\ 0 & 0 & -\nu\varepsilon_{11} \end{vmatrix}$$

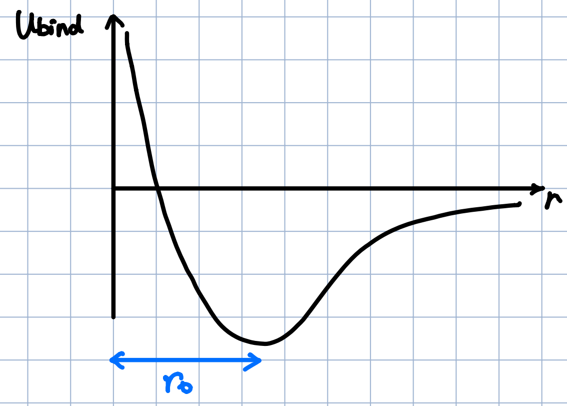



plasticity $\nu = \frac{1}{2}$

# Ideal or theoretical shear strength




constant

$$\tau \sim k \cdot \frac{\sin 2\pi x}{r_0}$$

for small x, $\sin 2\pi x / r_0 \sim 2\pi x / r_0$

$$\tau = \frac{k 2\pi x}{r_0}$$

$$\gamma = 2\varepsilon_{12} \sim x/d \longrightarrow \tau = G\gamma \sim Gx/d$$

$$\rightarrow Gx/d \sim \frac{k 2\pi x}{r_0} \rightarrow k \sim Gr_0/2\pi d$$

$$\therefore \tau = Gr_0/2\pi d \cdot \frac{\sin 2\pi x}{r_0}$$ Ans.

$\tau_{max} = G/2\pi$ when $\frac{\sin 2\pi x}{r_0} = 1$

$d \sim r_0$ (order)

$y_{Th} = 2\tau_{Th} \sim G/5$ : Theoretical strengths

2 orders of magnitude

$\rightarrow$ defects in materials

$\tau_{Th} \gg \tau_{exp}$ $\because$ dislocations ★★

$\tau_{Th}^{cohesive} > \tau_F$