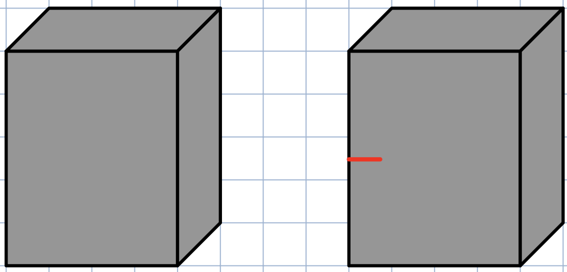
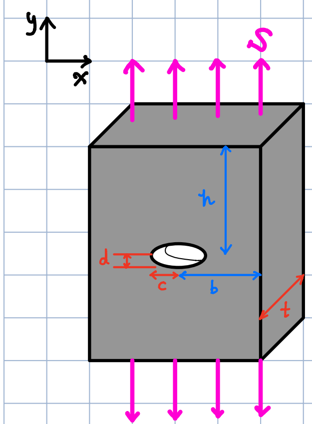
# Fracture Mechanics



The presence of a crack → fails below $\sigma_y$

why? Fracture Mechanics!

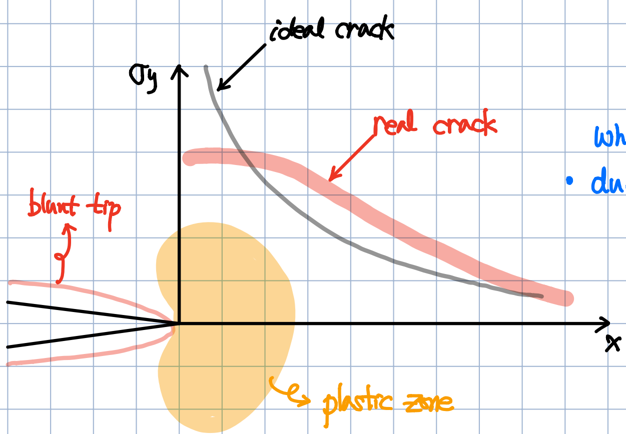
## Crack as Stress Raiser




$\sigma_y = S\,(1+2\,c/d)$ ← stress concentration

$= S\,(1+2\sqrt{c/\rho})$ ← tip radius, $\rho = d^2/c$

∴ stress concentration factor : $\boxed{k_t = \sigma_y / S}$




Why?

- ductile material → large plastic deformation

- the crack tip → $\sigma \uparrow$ → $\sigma > \sigma_y$ → plastic zone → sharp tip becomes blunt → $\sigma = \infty$ ✗

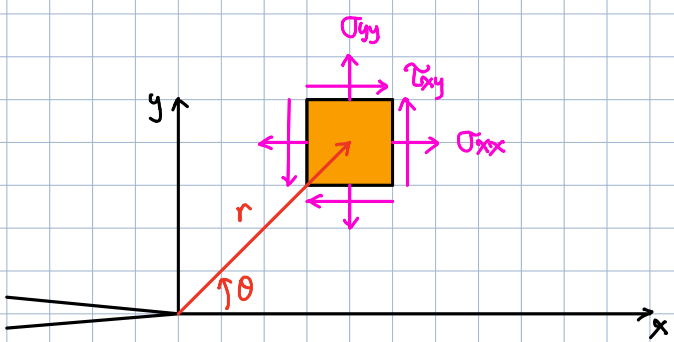
# Stress Analysis of Crack (Stress Intensity Factor, K)

- For certain cracked configuration
- under external forces
- isotropic
- linear elastic
- a polar coordinate axes (origin is at the crack tip)

$$\boxed{\sigma_{ij} = \left(\frac{k}{\sqrt{r}}\right) f_{ij}(\theta) + \sum_{m=0}^{\infty} A_m \cdot r^{m/2} \cdot g_{ij}^{(m)}(\theta)}$$

← Westergaard, Sneddon, Williams

$k$ : a constant
$f_{ij}$ : a dimensionless function of $\theta$




Each mode of loading produces the $1/\sqrt{r}$ singularity at the crack tip

$k$ & $f_{ij}$ → depend on the mode
→ $K_I, K_{II}, K_{III}$

$K = k\sqrt{2\pi}$
(the stress intensity factor)

## Mode I

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \cdot \left(1 - \sin\frac{\theta}{2} \cdot \sin\frac{3\theta}{2}\right) + \cdots$$

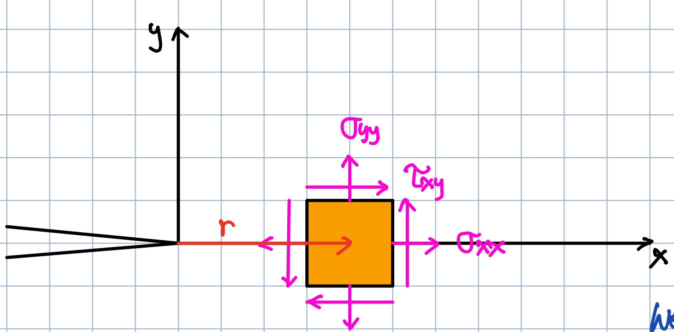
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \cdot \left(1 + \sin\frac{\theta}{2} \cdot \sin\frac{3\theta}{2}\right) + \cdots$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \cdot \left(\sin\frac{\theta}{2} \cdot \sin\frac{3\theta}{2}\right) + \cdots$$

$\sigma_{zz} = 0$ (plane stress)

$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ (plane strain ; $\varepsilon_{zz} = 0$)

$\tau_{yz} = \tau_{zx} = 0$




(on the crack plane)
$\theta = 0$ ;

$$\boxed{\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}}$$

$$\boxed{K_I = \lim_{r,\theta \to 0} \sigma_y \cdot \sqrt{2\pi r} = FS\sqrt{\pi a}}$$

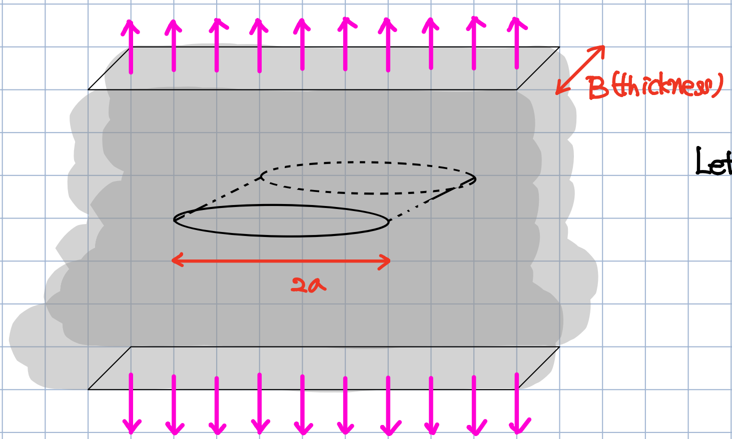
Westergaard solution
Anderson's textbook
Appendix 2A.3.2

Stress intensity has units of stress $\sqrt{\text{length}}$

## Griffith Stress

Assumption : The elastic energy in the elliptical area is released, when the crack with length 2a is introduced.

infinite plate




Let's consider a crack of length 2a in an "infinite" plate.

Total Energy (E) = Potential Energy (Π) + Work to create NEW SURFACE ($W_s$)

$$E = \Pi + W_s$$

$$dE/dA = d\Pi/dA + dW_s/dA$$

$\pi = U - F$

$= \frac{P\Delta}{2} - P\Delta$

$= -\frac{P\Delta}{2} = -U$

$d\Pi/dA = -dU/dA$

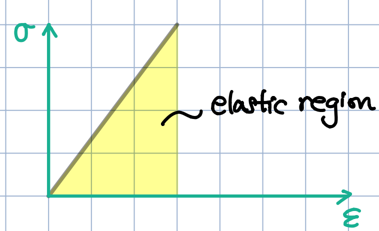
$\gamma_s \sim 1\,J/m^2$, surface energy

$\gamma_s \cdot 2aB \times 2 = 4aB\gamma_s$

F : the work done by external forces

$U_\varepsilon$ : the strain energy

$U_\varepsilon = \sigma^2/2E \cdot$ Volume $= \sigma^2/2E \cdot$




$u = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\sigma \cdot (\sigma/E) = \sigma^2/2E \cdot (\pi a^2 \cdot B)$

$\therefore U_\varepsilon = \sigma^2/2E \cdot$ Volume

| | | |
|---|---|---|
| $A = 2aB$ | $U_\varepsilon = \sigma^2/2E \cdot (\pi a^2 B)$ | $W_s = 4aB \cdot \gamma_s$ |
| $dA/da = 2B$ | $dU_\varepsilon/da = \sigma^2/2E \cdot (2\pi aB)$ | $dW_s/da = 4B \cdot \gamma_s$ |

$\therefore dU_\varepsilon/dA = \frac{dU_\varepsilon/da}{dA/da} = \sigma^2/E\,(\pi a)$ , $dW_s/da = 2\gamma_s$

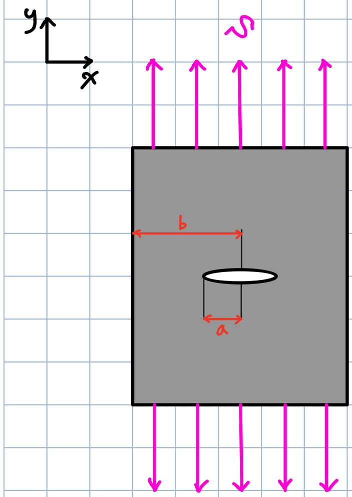
$$dE/dA = d\Pi/dA + dW_s/dA = -\sigma^2/E\,(\pi a) + 2\gamma_s = 0 \quad (\because \text{Energy Balance}) \longrightarrow \sigma^2 = \frac{2E\gamma_s}{\pi a}$$

$$\therefore \sigma_f = \sqrt{\frac{2E\gamma_s}{\pi a}}$$

Ans.

# The fracture toughness, $K_c$

1. $K$ : the stress intensity factor (LEFM)
2. $K_c$ : the fracture toughness (vary widely)
3. $K_{Ic}$ : the plane strain fracture toughness (material's ability)




$$\boxed{K = S\sqrt{\pi a}} \quad (a \ll b)$$

For a given material and thickness with $K_c$

$$\boxed{S_c = \frac{K_c}{\sqrt{\pi a}}}$$

$\downarrow$

$$\pi a_t = \left(\frac{K_c}{\sigma_o}\right)^2$$

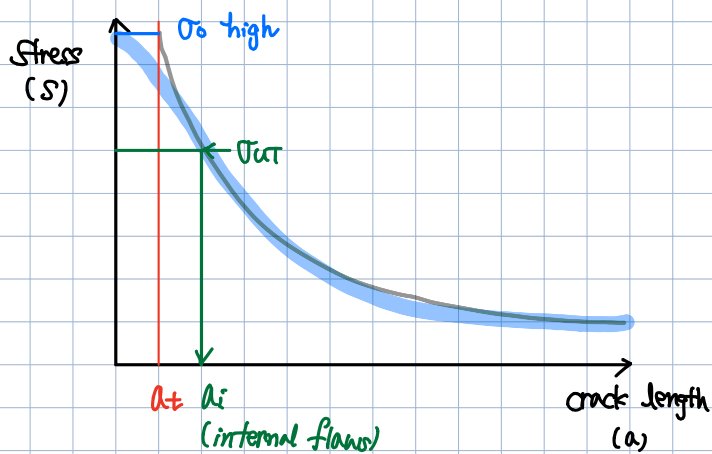
$$\boxed{\therefore \; a_t = \frac{1}{\pi}\cdot\left(\frac{K_c}{\sigma_o}\right)^2}$$

$a_t$ = the transition crack length

my crack size $> a_t$

→ brittle fracture rather yielding.

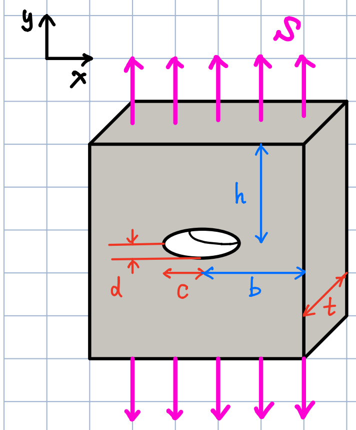

Polymer




Ceramics

Example :

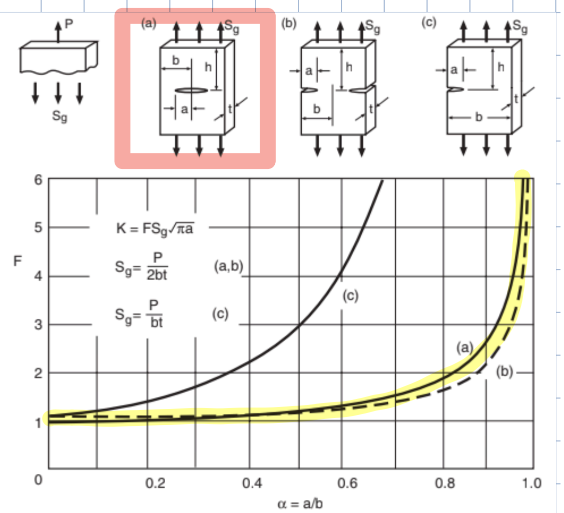
A center-cracked plate




$b = 50$ mm
$t = 5$ mm
large $h$

$P = 50$ kN

tip radius, $\rho = d^2/c$




**Values for small $a/b$ and limits for 10% accuracy:**

| (a) $K = S_g\sqrt{\pi a}$ | (b) $K = 1.12 S_g\sqrt{\pi a}$ | (c) $K = 1.12 S_g\sqrt{\pi a}$ |
|---|---|---|
| $(a/b \le 0.4)$ | $(a/b \le 0.6)$ | $(a/b \le 0.13)$ |

**Expressions for any $\alpha = a/b$:**

(a) $F = \dfrac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}}$ $\quad (h/b \ge 1.5)$

(b) $F = \left(1 + 0.122\cos^4\dfrac{\pi\alpha}{2}\right)\sqrt{\dfrac{2}{\pi\alpha}\tan\dfrac{\pi\alpha}{2}}$ $\quad (h/b \ge 2)$

(c) $F = 0.265(1-\alpha)^4 + \dfrac{0.857 + 0.265\alpha}{(1-\alpha)^{3/2}}$ $\quad (h/b \ge 1)$

a) $K = ?$ ← when $a = 10$ mm
(the stress intensity factor)

b) $a = 30$ mm

$K_{IC} = 24\text{MPa}\sqrt{m}$

c) $a_c$ (for 2014-T651 Aluminum)

F is a dimensionless function that depends on the geometry and loading configuration

a) $K = FS_g\sqrt{\pi a}$ ← $S_g = \dfrac{P}{2bt} = \dfrac{(50\text{kN})}{(2)\cdot(50\text{mm})\cdot(5\text{mm})} = \underline{100\text{ MPa}}$

$\alpha = a/b = \dfrac{10\text{mm}}{50\text{ mm}} = 0.2 < 0.4 \longrightarrow \underline{F = 1}$

$K = (1)\cdot(100\text{MPa})\cdot\sqrt{(\pi)\cdot(10\text{mm})} = \underline{17.7\text{ MPa}\sqrt{m}}$ Ans.

b) $K = FS_g\sqrt{\pi a}$ ← $S_g = 100\text{MPa}$

$\alpha = a/b = \dfrac{30\text{ mm}}{50\text{ mm}} = \underline{0.6} > 0.4 \longrightarrow F = \dfrac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}}$

$= \underline{1.292}$

$\therefore K = (1.292)\cdot(100\text{MPa})\cdot\sqrt{\pi\cdot(30\text{mm})} = \underline{39.7\text{ MPa}\sqrt{m}}$ Ans.

c) Table $\rightarrow$ 2014-T651 Aluminum : $K_{Ic} = 24 MPa\sqrt{m}$

We don't know $a$ $\longrightarrow$ assume that $\alpha = a/b < 0.4$ $\rightarrow$ $F = 1$ $\rightarrow$ $K = S_g\sqrt{\pi a}$

$K/S_g = \sqrt{\pi a}$

with the 10% approximation

$$a = 1/\pi \cdot (K/S_g)^2 = 1/\pi \cdot \left(\frac{24 MPa\sqrt{m}}{100 MPa}\right)^2 = 0.0183 m = \underline{18.3 mm}$$ Ans.,

Double check! $\alpha = a/b = \frac{18.3 mm}{50 mm} = 0.37 < 0.4 \longrightarrow \underline{F = 1}$ okay!

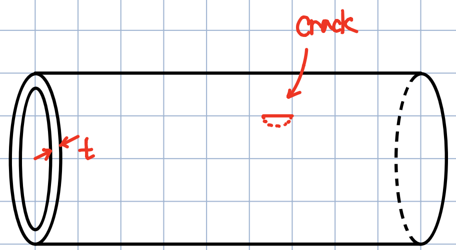
Example>

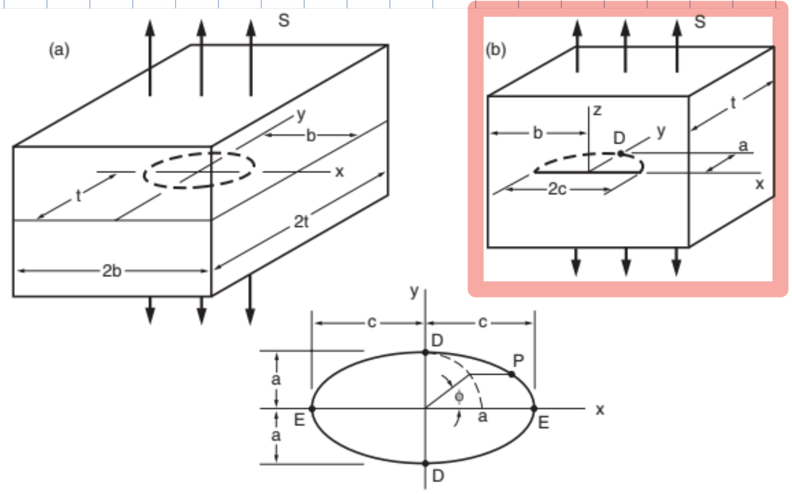
A pressure vessel (ASTM A517-F steel)
RT
$t = 50\,mm$

$K_{Ic} = 187\ MPa\sqrt{m}$

$\sigma_0 = 760\ MPa$

A surface crack : A semi-elliptical shape
surface length : $2c = 40\,mm$
depth = $a = 10\,mm$
$S_z = 300\,MPa$ (normal to the crack plane)
$S_x = 150\,MPa$ (parallel to the crack plane)




$$K_D = F_D S\sqrt{\frac{\pi a}{Q}},$$

$$Q \approx 1 + 1.464\left(\frac{a}{c}\right)^{1.65} \qquad (a/c \le 1)$$

| Case | Values for small $a/t$, $c/b$ | Limits for 10% accuracy |
|---|---|---|
| (a) | $F_D = 1$ | $a/t < 0.4$, $c/b < 0.2$ |
| (b) | $F_D \approx 1.12$ | $a/t < 0.3$,[1] $c/b < 0.2$ |

Note: [1]Except limit to $a/t < 0.16$ if $a/c < 0.25$.

**Figure 8.19** Stress intensity factors for (a) an embedded elliptical crack and (b) a similar half-elliptical surface crack. The equations give $K_D$ at point $D$ for a uniform tension normal to the crack plane. (Based on [Newman 86].)

Safety Factor ?

$$\boxed{SF = \frac{K_{Ic}}{K}} = \frac{187\ MPa\sqrt{m}}{49.2\ MPa\sqrt{m}} = 3.80 \quad \text{Ans.}$$

$$K = K_D = F_D S\sqrt{\pi a / Q} \leftarrow$$

$a/c = 10\,mm / 20\,mm = 0.5 < 1$

$Q = 1 + (1.464)(a/c)^{1.65} = 1.466$ ★

$a/t = 10\,mm / 50\,mm = 0.2 < 0.3$ $\quad \therefore F_D \approx 1.12$ ★★

$$\therefore K = (1.12)\cdot(300\,MPa)\cdot\sqrt{\frac{(\pi)\cdot(0.010\,m)}{1.466}} = 49.2\ MPa\sqrt{m}$$

Example >

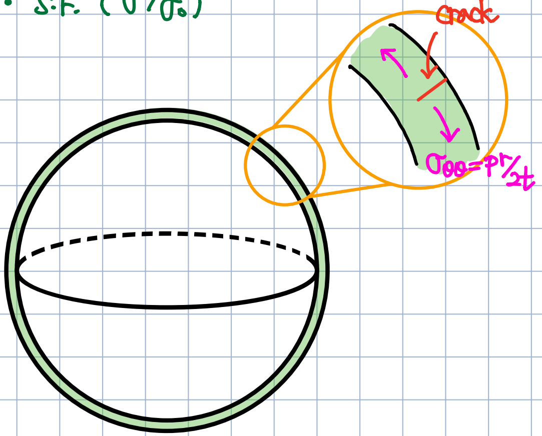
A spherical pressure vessel (ASTM A517-F steel)

$d_{in} = 1.5\,m$ $\qquad$ $K_{Ic} = 187\ MPa\sqrt{m}$

$t = 10\,mm$ $\qquad$ $\sigma_0 = 760\ MPa$

$p = 6\,MPa$

- the leak-before-break condition ?
- S.F. ( $K/K_{Ic}$ )
- S.F. ( $\sigma/\sigma_0$ )




Leak-Before-Break Design

- a thin-walled pressure vessel with a crack growing

1. The crack will gradually extend and penetrate wall
2. Sudden brittle fracture (before leaking)
   ↳ explosive release

$$\sigma_t = \frac{pr}{2t} = \frac{(6MPa)\cdot(750mm)}{(2)\cdot(10mm)} = \underline{225MPa}$$

$K = FS\sqrt{\pi a}$ ← $F = 1$

↓

$$\boxed{a_c = \frac{1}{\pi}\cdot\left(\frac{K_{Ic}}{\sigma_t}\right)^2} = \frac{1}{\pi}\cdot\left(\frac{187MPa\sqrt{m}}{225MPa}\right)^2 = 0.220\,m = \underline{220\,mm}$$ Ans.

the leak-before-break condition is met.

if the vessel leaks,

$$K = FS\sqrt{\pi a} = (1)\cdot(225MPa)\cdot\sqrt{\pi\cdot(0.01m)} = 39.9\,MPa\sqrt{m}$$

$$\therefore SF = \frac{K_{Ic}}{K} = \frac{187MPa\sqrt{m}}{39.9MPa\sqrt{m}} = \underline{4.69}$$ Ans.

$$SF = \sigma_0/\sigma_t = \frac{760MPa}{225MPa} = \underline{3.38}$$ Ans.